# Renormalization-Group Solutions for Yukawa Potential

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The self–similar renormalization group is used to obtain expressions for the spectrum of the Hamiltonian with the Yukawa potential. The critical screening parameter above which there are no bound states is also obtained by this method. The approach presented illustrates that one can achieve good accuracy without involving extensive numerical calculations, but invoking instead the renormalization–group techniques.

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#### I. INTRODUCTION

Renormalization—group techniques are widely used in quantum field theory, statistical mechanics, and solid—state physics. Their usage in atomic physics is less customary. The aim of the present paper is to show how renormalization—group ideas can be applied for calculating the spectra of quantum—mechanical Hamiltonians with realistic potentials.

As a model for illustration we opt for a Hamiltonian with the Yukawa potential. This choice is based on the special role of this potential in different branches of physics. In plasma physics it is known as the Debye–Hückel potential, in solid–state physics and atomic physics it is called the Thomas–Fermi or screened Coulomb potential, and in nuclear physics one terms it the Yukawa potential. Among recent important applications of this potential we may mention its use in the models describing metal–insulator transition [1].

The problem of finding the energy levels for the Yukawa potential has received a lot of attention in literature. Several approaches have been used for solving this problem: the simple variational procedure [2], use of atomic orbitals with a set of fitting parameters [3], Rayleigh–Schrödinger perturbation theory [4,5], method of potential envelopes [6-9], an iterative procedure [10], and different numerical calculations [11-14].

In this paper we demonstrate how the problem can be treated by employing the self–similar approximation theory [15-19] based on the ideas of the renormalization–group and dynamical theory. The outline of the paper is as follows. In Sec.II we sketch the main steps of the procedure using the self–similar renormalization group. We present only those formulas that are necessary for understanding the following calculations; all details and mathematical foundation can be found in Refs.[15-19]. In Sec.III we apply the approach to the Schrödinger equation with the Yukawa potential. In Sec.IV we obtain the sequence of renormalized energies. The convergence of this sequence is governed by control functions defined from the minimum of multiplier. In Sec.V the procedure is applied to calculating the renormalized critical screening parameter. Emphasize that we are presenting here an analytical method, not relying on heavy numerical calculations. Despite its analytical nature, the method gives quite good accuracy for the found critical screening parameter.

## II. SELF-SIMILAR RENORMALIZATION

We give here the general sketch of the procedure [15-19], without specifying the nature of the functions involved. Suppose a function f(x) is defined by a complicated equation that is being solved approximately. Employing a perturbative algorithm, we may get a sequence of approximations,  $\{F_k\}_{k=0}^{\infty}$ , for the sought function f(x). To make the sequence convergent, we incorporate into the approximations  $F_k = F_k(x, u_k)$ , with  $k = 0, 1, 2, \ldots$ , a set of control functions  $u_k = u_k(x)$ , so that the sequence  $\{f_k(x)\}_{k=0}^{\infty}$  of the approximations

$$f_k(x) \equiv F_k(x, u_k(x)) \tag{1}$$

be convergent.

Make the change of variables by defining a function  $x_k(f)$  through the equation

$$F_0(x, u_k(x)) = f, \qquad x = x_k(f),$$
 (2)

in which f is the new variable. With this change of variables, Eq.(1) yields

$$y_k(f) \equiv f_k(x_k(f)). \tag{3}$$

The transformation inverse to Eq.(3) is

$$f_k(x) = y_k(F_0(x, u_k(x))).$$
 (4)

Construct an approximation cascade  $\{y_k\}$  by requiring the self-similarity relation

$$y_{k+p}(f) = y_k(y_p(f)). \tag{5}$$

The trajectory  $\{y_k(f)\}_{k=0}^{\infty}$  of this approximation cascade is, according to Eqs.(3) and (4), bijective to the sequence  $\{f_k(x)\}_{k=0}^{\infty}$  of approximations in Eq.(1). Embedding the approximation cascade into an approximation flow and integrating the evolution equation for the latter, we obtain the evolution integral

$$\int_{f_k}^{f_{k+1}^*} \frac{df}{v_k(f)} = t_k^*,\tag{6}$$

in which  $f_k = f_k(x)$  is a given approximation,  $f_k^* = f_k^*(x)$  is a renormalized self–similar approximation, and

$$v_k(f) = F_{k+1}(x_k, u_k) - F_k(x_k, u_k) + (u_{k+1} - u_k) \frac{\partial}{\partial u_k} F_k(x_k, u_k)$$
(7)

is the cascade velocity, where  $x_k = x_k(f)$  and  $u_k = u_k(x_k(f))$ . The right-hand side of Eq.(6), that is  $t_k^*$ , is the minimal time necessary for reaching the renormalized approximation  $f_{k+1}^*(x)$ .

A fixed point  $y_k^*(f)$  of the approximation cascade represents, by construction, the sought function f(x) which can be obtained from transformation (4). At the fixed point the cascade velocity  $v_k(f) \to 0$  as  $k \to \infty$ . If the cascade velocity is not zero exactly, but only approximately, then we have not an actual fixed point, but a quasifixed point. For instance, assuming that  $v_k(f) \approx 0$  and  $F_{k+1} \approx F_k$ , from Eq.(7) we have

$$(u_{k+1} - u_k) \frac{\partial}{\partial u_k} F_k(x, u_k) = 0.$$
(8)

which is a quasifixed-point condition.

The convergence of the approximation sequence  $\{f_k(x)\}_{k=0}^{\infty}$  is equivalent to the stability of the cascade trajectory  $\{y_k(f)\}_{k=0}^{\infty}$ . The stability of the latter can be analysed by defining the multipliers

$$\mu_k(f) \equiv \frac{\partial}{\partial f} y_k(f) \tag{9}$$

and

$$m_k(x) \equiv \frac{\delta F_k(x, u_k(x))}{\delta F_0(x, u_k(x))}.$$
(10)

These multipliers are images of each other being related by the transformations

$$\mu_k(f) = m_k(x_k(f)), \qquad m_k(x) = \mu_k(F_0(x, u_k(x))).$$
 (11)

The trajectory is locally stable when

$$|m_k(x)| \le 1, \qquad |\mu_k(f)| \le 1.$$
 (12)

The multipliers (9) and (10) describe the local stability, at the step k, of the cascade trajectory with respect to the variation of initial conditions. This type of local multipliers can be called quasilocal multipliers [19]. Another type of local multipliers defined as

$$m_k^*(x) \equiv \frac{m_k(x)}{m_{k-1}(x)}$$

characterizes the local stability, at the step k, with respect to the variation of the point k-1. The latter multipliers can be termed ultra-local.

Recall that control functions are introduced so that to provide the convergence of the approximation sequence  $\{f_k(x)\}_{k=0}^{\infty}$ , that is, the stability of the cascade trajectory  $\{y_k(f)\}_{k=0}^{\infty}$ . This suggests a way for the practical definition of control functions. The local multiplier (10) may be written as

$$m_k(x) = M_k(x, u_k(x)) \tag{13}$$

with

$$M_k(x,u) = \frac{\partial F_k(x,u)}{\partial u} / \frac{\partial F_0(x,u)}{\partial u}.$$
 (14)

To produce the maximal stability of the cascade trajectory, i.e., the fastest convergence of the approximation sequence, for each fixed value of x, we have to require the minimum of the absolute value for the multiplier (13) with respect to the control function  $u_k(x)$ . In other words, the principal of maximal stability is the condition for the minimum of the multiplier modulus,

$$\min_{u} |M_k(x, u)| = |M_k(x, u_k(x))|. \tag{15}$$

This condition gives us a constructive definition of control functions.

## III. YUKAWA POTENTIAL

Now we illustrate how the scheme of Sec.II applies for calculating the eigenvalues of the radial Hamiltonian

$$H = -\frac{1}{2m}\frac{d^2}{dr^2} + \frac{l(l+1)}{2mr^2} - \frac{A}{r}e^{-\alpha r}$$

with the Yukawa potential.

It is convenient to pass to dimensionless quantities by scaling the above Hamiltonian and reducing it to the form

$$H = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} - \frac{e^{-\alpha r}}{r}.$$
 (16)

Here  $r \in [0, \infty)$ ;  $l = 0, 1, 2, \ldots$ ; and  $\alpha$  is a positive constant called the screening parameter. To return back to dimensional quantities one has to make the following substitutions:

$$r \to mAr, \qquad \alpha \to \frac{\alpha}{mA}, \qquad H \to \frac{H}{mA^2}.$$

Write Hamiltonian (16) as the sum  $H \equiv H_0 + \Delta H$ , with the first term being the Hamiltonian

$$H_0 = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} - \frac{u}{r} \tag{17}$$

with a Coulomb–type potential, where u is yet unknown quantity which will latter generate control functions. Employing some variant of perturbation theory in powers of the perturbation

$$\Delta H \equiv H - H_0 = \frac{u - e^{-\alpha r}}{r},\tag{18}$$

we may construct a sequence of approximate eigenvalues,  $E_k$ , and eigenfunctions,  $\psi_k$ , respectively,

$$E_k \equiv E_{nl}^{(k)}(\alpha, u), \qquad \psi_k \equiv \psi_{nl}^{(k)}(r, u), \tag{19}$$

where k = 0, 1, 2, ... enumerates approximations, while n = 0, 1, 2, ... and l = 0, 1, 2, ... are the quantum numbers labelling the energy levels. For the initial approximation one has the spectrum

$$E_0 = -\frac{u^2}{2(n+l+1)^2},\tag{20}$$

and the wave function

$$\psi_0 = \left[ \frac{n!u}{(n+2l+1)!} \right]^{1/2} \frac{1}{n+l+1} \left( \frac{2ur}{n+l+1} \right)^{l+1} \exp\left( -\frac{ur}{n+l+1} \right) L_n^{2l+1} \left( \frac{2ur}{n+l+1} \right), \tag{21}$$

in which

$$L_n^l(r) = \sum_{m=0}^{n} \frac{\Gamma(n+l+1)(-r)^m}{\Gamma(m+l+1)(n-m)!m!}$$

is an associate Laguerre polynomial.

To find the subsequent approximations, we could use the Rayleigh–Schrödinger perturbation theory. However this would involve the following complication. The whole spectrum of the Hamiltonian (16) contains, in addition to discrete levels, the continuous part. Therefore, we would have to deal, besides the summation over discrete levels, with the integration over continuous ones.

To avoid this complication, when we are interested only in discrete levels, we may employ the Dalgarno–Lewis perturbation theory [20]. Then, for the k-approximation one writes

$$E_k = E_0 + \sum_{p=1}^k \Delta E_p,$$

$$\psi_k = \psi_0 + \sum_{p=1}^k \Delta \psi_p. \tag{22}$$

The first correction for the eigenvalues is

$$\Delta E_1 = (\psi_0, \Delta H \psi_0), \tag{23}$$

and the first correction to the eigenfunction is a solution to the equation

$$(H_0 - E_0)\Delta\psi_1 = (\Delta E_1 - \Delta H)\psi_0. \tag{24}$$

Solving the Dalgarno equation (24), one may calculate the second correction to the eigenvalues,

$$\Delta E_2 = (\Delta \psi_1, \Delta H \psi_0), \tag{25}$$

and so on.

The Dalgarno equation (24) is a nonhomogeneous equation whose solution can be written as the sum of the general solution to the corresponding homogeneous equation plus a particular solution to the nonhomogeneous equation. The solution to the homogeneous equation is, as is evident, proportional to  $\psi_0$ . So we may set

$$\Delta \psi_1 = C\psi_0 + \varphi,\tag{26}$$

with the proportionality constant C defined by the normalization condition  $(\psi_k, \psi_k) = 1$ , and the function  $\varphi$  being a particular solution to the nonhomogeneous equation

$$(H_0 - E_0)\varphi = (\Delta E_1 - \Delta H)\psi_0. \tag{27}$$

From the normalization condition  $(\psi_1, \psi_1) = 1$ , for the function  $\psi_1 = \psi_0 + \Delta \psi_1$ , omitting the second-order term, one has

$$(\psi_0, \Delta \psi_1) = 0. \tag{28}$$

With Eq.(26), this gives

$$C = -(\psi_0, \varphi). \tag{29}$$

Following the scheme described, with the notation

$$\beta \equiv \frac{\alpha}{2u}(n+l+1),\tag{30}$$

we obtain the first correction for the eigenvalues of bound states,

$$\Delta E_1 = \frac{u^2 I_0 - u I_\beta}{(n+l+1)^2},\tag{31}$$

where the integral

$$I_{\beta} \equiv \frac{n!}{(n+2l+1)!} \int_{0}^{\infty} r^{2l+1} e^{-(1+\beta)r} \left[ L_{n}^{2l+1}(r) \right]^{2} dr$$

can be expressed as

$$I_{\beta} = \frac{(\beta - 1)^n}{(\beta + 1)^{n+2l+2}} \mathcal{P}_n^{2l,0} \left( \frac{\beta^2 + 1}{\beta^2 - 1} \right)$$

through the Jacobi polynomials

$$\mathcal{P}_n^{k,p}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-k} (1+x)^{-p} \frac{d^n}{dx^n} \left[ (1-x)^{n+k} (1+x)^{n+p} \right] =$$

$$= \frac{1}{2^n} \sum_{m=0}^n C_{n+k}^m C_{n+p}^{n-m} (x-1)^{n-m} (x+1)^m$$

having the properties

$$\mathcal{P}_n^{k,p}(1) = C_{n+k}^n, \qquad \mathcal{P}_n^{k,p}(-1) = (-1)^n C_{n+p}^n, \qquad C_n^m \equiv \frac{n!}{(n-m)!m!}.$$

Another integral in Eq.(31) is

$$I_0 \equiv \lim_{\beta \to 0} I_{\beta} = 1.$$

In this way, the first approximation  $E_1 = E_0 + \Delta E_1$  becomes

$$E_1 = \frac{u^2 - 2uI_\beta}{2(n+l+1)^2},\tag{32}$$

where

$$I_{\beta} = \frac{1}{(1+\beta)^{2n+2l+2}} \sum_{m=0}^{n} C_{n+2l+1}^{m} C_{n}^{m-m} \beta^{2m}.$$

For the ground–state level, when n = l = 0 and  $\beta = \alpha/2u$ , Eq.(32) reduces to

$$E_1 = -u^2 \left(\frac{1}{2} - \sigma\right),\tag{33}$$

where the notation

$$\sigma \equiv 1 - \frac{4u}{(2u + \alpha)^2} \tag{34}$$

is introduced.

The ground state plays a special role defining, when it becomes zero, the critical screening parameter  $\alpha_c$ , above which there are no bound states. Therefore, in what follows we consider the ground state.

Eq.(27), with

$$E_0 = -\frac{u^2}{2}, \qquad \psi_0 = 2u^{3/2}re^{-ur}, \tag{35}$$

writes

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} - \frac{u}{r} + \frac{u^2}{2}\right)\varphi = \left(\sigma u^2 - \frac{u}{r} + \frac{e^{-\alpha r}}{r}\right)\psi_0. \tag{36}$$

The solution to Eq.(36) must be a bounded function,  $|\varphi(r)| < \infty$ , for all  $r \in [0, \infty)$ . Let us present  $\varphi$  as the product

$$\varphi(r) = \psi_0(r)g(ur),$$

in which the second factor satisfies the equation

$$\frac{d}{dr}g(r) = \rho(r)\frac{e^{2r}}{r^2}$$

where  $\rho(r)$  is to be defined from Eq.(36), which yields

$$\frac{d\rho}{dr} = (1 - \sigma r)2re^{-2r} - \frac{2r}{u}\exp\left(-\frac{2u + \alpha}{u}r\right).$$

The latter equation gives

$$\rho(r) = \left[\sigma r^2 + (\sigma - 1)\left(r + \frac{1}{2}\right)\right]e^{-2r} + \frac{1 - \sigma}{2}\left(1 + \frac{2u + \alpha}{u}r\right)\exp\left(-\frac{2u + \alpha}{u}r\right) + C_1,$$

with an integration constant  $C_1$ . The equation for g(r) results in

$$g(ur) = \sigma ur + (1 - \sigma) \left[ \frac{1 - e^{-\alpha r}}{2ur} - \ln(ur) + Ei(-\alpha r) \right] + C_1 \left[ 2Ei(2ur) - \frac{e^{2ur}}{ur} \right] + C_2,$$

where  $C_2$  is an integration constant and

$$Ei(ar) = \int_{-\infty}^{r} \frac{e^{ax}}{x} dx$$

is the exponential-integral function.

The boundness of  $\varphi$  requires that  $C_1 = 0$ . The additive term containing a function proportional to  $\psi_0$  should be omitted in  $\varphi$ , since such a term has already been included into Eq.(26). The latter means that we have to put  $C_2 = 0$ . As a result we obtain

$$\varphi(r) = \sqrt{u}e^{-ur}\left\{2\sigma(ur)^2 + (1-\sigma)\left[1 - e^{-\alpha r} - 2ur\ln(ur) + 2urEi(-\alpha r)\right]\right\}. \tag{37}$$

For the normalization constant in Eq.(29) we find

$$C = -\frac{3}{2}\sigma + (1 - \sigma)\left(\frac{\alpha}{2u + \alpha} - \ln\frac{2\alpha}{2u + \alpha} - \gamma_E\right)$$
(38)

with the Euler constant  $\gamma_E = 0.577215$ .

After finding function (26), we can calculate the second correction (25), which yields

$$\Delta E_2 = u^2 J_1 - u J_2, \tag{39}$$

where

$$J_1 = \lim_{\nu \to 0} J_2(\nu), \qquad J_2 = J_2(\beta),$$

with  $\beta = \alpha/2u$ , and

$$J_2(\nu) = 2 \int_0^\infty \left\{ 2Cr + 2\sigma r^2 + (1 - \sigma) \left[ 1 - e^{-\alpha r/u} - 2r \ln r + 2r Ei \left( -\frac{\alpha r}{u} \right) \right] \right\} e^{-2(1+\nu)r} dr.$$

Thus, for the second approximation for the energy we obtain

$$E_2 = E_1 + u^2 J_1 - u J_2, (40)$$

where

$$J_1 = -\frac{\sigma}{2} + \frac{1-\sigma}{1+\beta}\beta,$$

$$J_2 = -u(1-\sigma)\frac{\sigma(1+3\beta)}{2(1+\beta)} + u(1-\sigma)^2 \left[ \frac{(1+3\beta+\beta^2)\beta}{(1+\beta)(1+2\beta)} + \ln\frac{(1+\beta)^2}{1+2\beta} \right].$$

### IV. RENORMALIZED ENERGY

From the results of the previous section we derive the following sequence of approximations for the energy:

$$E_0 = -\frac{u^2}{2},$$

$$E_1 = E_0 + u^2 - \frac{4u^3}{(2u+\alpha)^2},$$

$$E_{2} = E_{1} - \frac{u^{2}}{2} + \frac{2u^{3}}{(2u+\alpha)^{2}} + \frac{4u^{3}\alpha}{(2u+\alpha)^{3}} + \frac{2u^{3}(2u^{2} + 5u\alpha - 2\alpha + 3\alpha^{2})}{(u+\alpha)(2u+\alpha)^{3}} - \frac{8u^{4}(2u^{2} + 5u\alpha + 4\alpha^{2})}{(u+\alpha)(2u+\alpha)^{5}} - \frac{16u^{4}}{(2u+\alpha)^{4}} \ln \frac{(2u+\alpha)^{2}}{4u(u+\alpha)}.$$
(41)

For multiplier defined in Eq.(14) we have

$$M_k(\alpha, u) = \frac{\partial E_k(\alpha, u)}{\partial u} / \frac{\partial E_0(\alpha, u)}{\partial u}.$$
 (42)

Substituting here the derivatives

$$\frac{\partial E_0}{\partial u} = -u,$$

$$\frac{\partial E_1}{\partial u} = u - \frac{12u^2}{(2u+\alpha)^2} + \frac{16u^3}{(2u+\alpha)^3},$$

following from the sequence (41), we find

$$M_1(\alpha, u) = -\frac{8u^3 - 4(2 - 3\alpha)u^2 - 6\alpha(2 - \alpha)u + \alpha^3}{(2u + \alpha)^3}.$$
(43)

The control function  $u(\alpha) = u_1(\alpha)$  is to be defined from the principle of maximal stability (15). To this end, we, first, try the equation

$$M_1(\alpha, u) = 0, (44)$$

which gives

$$8u^3 - 4(2 - 3\alpha)u^2 - 6\alpha(2 - \alpha)u + \alpha^3 = 0.$$
(45)

This cubic equation has three roots of which we have to select a real one satisfying the asymptotic condition

$$\lim_{\alpha \to 0} u(\alpha) = 1.$$

The latter implies that if the screening parameter tends to zero, so that the Yukawa potential transforms into the Coulomb one, then one must return to the exact solution known for the Coulomb potential. Really, under  $\alpha \to 0$  and  $u \to 1$ , from the sequence in Eq.(41) it follows that  $E_k \to -\frac{1}{2}$  for all k. Eq.(45), with this asymptotic condition, yields the control function

$$u(\alpha) = \frac{1}{3} - \frac{\alpha}{2} + \frac{2}{3}\sqrt{1 + \frac{3}{2}\alpha} \cos\frac{\varphi}{3},\tag{46}$$

in which

$$\varphi = \left\{ \begin{array}{ll} \varphi^*, & \varphi^* \ge 0, \ 0 \le \alpha \le \frac{3 + \sqrt{57}}{18}, \\ \pi - \varphi^*, \ \varphi^* \le 0, \ \frac{3 + \sqrt{57}}{18} \le \alpha \le \alpha_0, \end{array} \right.$$

where

$$\varphi^* = \arctan \frac{3\alpha\sqrt{3(3+20\alpha-27\alpha^2)}}{4+9\alpha-27\alpha^2},$$

and the upper value of the screening parameter, below which Eq.(45) possesses yet a solution, is

$$\alpha_0 \equiv \frac{10 + 7\sqrt{7}}{27} = 1.056306. \tag{47}$$

Thus, solution (46) exists only in the interval  $0 \le \alpha \le \alpha_0$ . For  $\alpha > \alpha_0$ , Eq.(45) has no real solutions satisfying the derived asymptotic condition.

For  $\alpha > \alpha_0$  we need to find a minimum of the multiplier (43), which is not necessarily zero. This can be done by solving the equation

$$\frac{\partial}{\partial u} M_1(\alpha, u) = 0, \tag{48}$$

which results in the control function

$$u(\alpha) = \left(\frac{\sqrt{7}}{2} - 1\right)\alpha = 0.322876\alpha. \tag{49}$$

One may notice that when Eq.(44) has a solution, then the procedure is similar to renormalizing perturbative terms by means of a variational optimization [21-26]. However, as is shown above, this equation not always possesses physically reasonable solutions. While the principle of maximal stability (15) always provides us with a solution defining a control function. Therefore, this principle is more general than the simple variational procedure.

Substituting the found control function into the approximations in the sequence (41), we get the renormalized expressions

$$e_k(\alpha) \equiv E_k(\alpha, u_k(\alpha)).$$
 (50)

For example, when  $\alpha \leq \alpha_0$ , we have

$$e_1(\alpha) = -\frac{u^2(2u - \alpha)}{2(2u + 3\alpha)},$$
 (51)

$$e_2(\alpha) = -\frac{u^2}{2} \left[ \frac{8u^4 + 16u^3\alpha - 2u^2\alpha^2 - 10u\alpha^3 + \alpha^4}{2u(u+\alpha)(2u+3\alpha)^2} + 2\left(\frac{2u+\alpha}{2u+3\alpha}\right)^2 \ln\frac{(2u+\alpha)^2}{4u(u+\alpha)} \right].$$
 (52)

Respectively, for  $\alpha > \alpha_0$ , we have to substitute the control function (49) into Eq.(41).

The self-similar approximation for the energy is to be defined from the evolution integral (6). When no additional constraints are imposed, the minimal number of steps for reaching a quasifixed point is, clearly, one,  $t_k^* = 1$ . In the interval  $0 \le \alpha \le \alpha_0$ , the cascade velocity, given by Eq.(7), is

$$v_1(f) = e_2(\alpha(f)) - e_1(\alpha(f)),$$
 (53)

where the function  $\alpha(f)$ , according to Eq.(2), is defined by the equation

$$E_0(\alpha, u(\alpha)) = -\frac{1}{2}u^2(\alpha) = f \tag{54}$$

resulting in  $\alpha = \alpha(f)$ . For the evolution integral (6), we have

$$\int_{e_1}^{e_2^*} \frac{df}{v_1(f)} = 1,\tag{55}$$

where  $e_1 = e_1(\alpha)$  and  $e_2^* = e_2^*(\alpha)$ .

We calculated the values of the renormalized energies  $e_1(\alpha)$ ,  $e_2(\alpha)$ , and  $e_2^*(\alpha)$  as functions of the screening parameter  $\alpha$ . To characterize the accuracy of these approximations, it is convenient to introduce the maximal percentage errors

$$\varepsilon_k \equiv \max_{\alpha} \left[ \frac{e_k(\alpha) - e(\alpha)}{|e(\alpha)|} \right] \times 100\%,$$

where  $e(\alpha)$  is an exact value of the energy. Notice that this definition of the maximal error has sense only when  $e(\alpha)$  is not close to zero. In our case this definition works for  $0 \le \alpha \le \alpha_0$ . For  $\alpha > \alpha_0$ , when  $e(\alpha) \to 0$ , it is possible to redefine the maximal error by shifting the definition of the energy by a constant [27].

The maximal percentage error, defined as is explained above, is 2% for  $e_1(\alpha)$  and for  $e_2(\alpha)$  and  $e_2^*(\alpha)$  it is 1%. The multipliers (10), with  $\alpha$  instead of x, satisfy the stability conditions of Sec.II. The stability of the procedure means its convergence which is also evident from the improvement of accuracy.

#### V. CRITICAL SCREENING

An important quantity characterizing the features of the Yukawa potential is the critical screening parameter, that is, such a value of the screening parameter  $\alpha = \alpha_c$  above which there are no bound states. This critical parameter is defined by the condition  $e(\alpha_c) = 0$ . For each approximation  $e_k(\alpha)$  for the ground–state energy there exists the corresponding critical parameter  $\alpha_k$  given by the equation

$$e_k(\alpha_k) = 0. (56)$$

From the approximations  $e_k(\alpha)$  obtained in the previous section, we find the sequence of approximations for the critical screening parameters:

$$\alpha_1 = 1, \qquad \alpha_2 = 1.0833. \tag{57}$$

To employ the self–similar renormalization for the sequence  $\{\alpha_k\}$ , as is described in Sec.II, we compose a sequence  $\{\alpha_k(\lambda)\}$  of the partial sums

$$\alpha_k(\lambda) = \sum_{i=1}^k (\alpha_i - \alpha_{i-1}) \lambda^{p_i}, \tag{58}$$

in which  $k \geq 1$  and  $\alpha_0(\lambda) \equiv 0$ . As is clear from Eq.(58),

$$\alpha_k = \lim_{\lambda \to 1} \alpha_k(\lambda). \tag{59}$$

Then the sequence of approximations

$$\alpha_1(\lambda) = \alpha_1 \lambda^{p_1},$$

$$\alpha_2(\lambda) = \alpha_1 \lambda^{p_1} + (\alpha_2 - \alpha_1) \lambda^{p_2}$$

can be renormalized in the way prescribed by Sec. II. First, we define the expansion function  $\lambda(f)$  by the equation

$$\alpha_1(\lambda) = \alpha_1 \lambda^{p_1} = f$$

which gives

$$\lambda(f) = \left(\frac{f}{\alpha_1}\right)^{1/p_1}.$$

Writing the cascade velocity as

$$v_1(f) = \alpha_2(\lambda(f)) - \alpha_1(\lambda(f)) = (\alpha_2 - \alpha_1)\lambda^{p_2}(f),$$

we come to the evolution integral

$$\int_{\alpha_1(\lambda)}^{\alpha_2^*(\lambda)} \frac{df}{(\alpha_2 - \alpha_1)\lambda^{p_2}(f)} = t^*.$$
(60)

Notice that  $\alpha_2^*(\lambda)$  in Eq.(60) depends also on the parameters

$$p \equiv p_1 \ge 0, \qquad q \equiv \frac{p_2}{p_1} - 1 \ge 0,$$
 (61)

so that we may write

$$\alpha_2^*(\lambda) = \alpha_2^*(\lambda, p, q). \tag{62}$$

Integrating Eq.(60), we obtain

$$\alpha_2^*(\lambda, p, q) = \left[\frac{\alpha_1^{1+q} \lambda^{pq}}{\alpha_1 - q(\alpha_2 - \alpha_1) \lambda^{pq} t^*}\right]^{1/q}.$$
(63)

To define p and q, consider the sequence  $\{\bar{y}_k(f)\}\$  of the terms  $\bar{y}_k(f) \equiv \alpha_k(\lambda(f))$ . Thus, we have

$$\bar{y}_1(f) = f,$$

$$\bar{y}_2(f) = f + (\alpha_2 - \alpha_1) \left(\frac{f}{\alpha_1}\right)^{q+1}.$$

For the latter sequence we can find the multipliers defined as in Eq.(14). As is obvious,  $M_1 = 1$  and

$$M_2(\lambda, p, q) = 1 + (\alpha_2 - \alpha_1)(1+q)\lambda^{pq}.$$
 (64)

The values of p and q are to be chosen so that to satisfy the principle of maximal stability (15), with p and q playing the role of control functions. Since, according to condition (59), we must put at the end  $\lambda \to 1$ , we can consider the multiplier (64) for  $\lambda \sim 1$ . In the case when  $\lambda > 1$ , the minimum of  $|M_2|$  from Eq.(64) is provided by q = 0. But if  $\lambda < 0$ , then this minimum can occur at q = 1. Here we keep in mind that q, given by Eq.(61), is an integer and that the difference  $\alpha_2 - \alpha_1$  is positive in agreement with Eq.(57). In this way, from expression (63) we derive

$$\alpha_2^*(\lambda, p, 0) = \alpha_1 \lambda^p \exp\left(\frac{\alpha_2 - \alpha_1}{\alpha_1} t^*\right), \tag{65}$$

if q = 0, and

$$\alpha_2^*(\lambda, p, 1) = \frac{\alpha_1^2 \lambda^p}{\alpha_1 - (\alpha_2 - \alpha_1) \lambda^p t^*},\tag{66}$$

when q = 1.

The effective time  $t^*$  has the meaning of the minimal number of steps providing the renormalization of  $\alpha_k$ , when  $\lambda \to 1$ . If we put  $\lambda \to 1$  in the evolution integral (60) before the integration, then for  $\alpha_2^*$  we would get  $\alpha_1 + (\alpha_2 - \alpha_1)t^*$ . From here we see that  $t^* = 0$  gives  $\alpha_1$ ; one step, that is  $t^* = 1$ , leads to  $\alpha_2$ ; and  $t^* = 2$  results in  $2\alpha_2 - \alpha_1$ . Therefore, the minimal number of steps necessary for getting a nontrivial renormalization is  $t^* = 2$ .

Eqs. (65) and (66) show that, when  $\lambda = 1$ , then  $\alpha_2^*$  does not depend on p. Consequently, we may write

$$\alpha_2^*(1,q) \equiv \alpha_2^*(1,p,q). \tag{67}$$

Putting in Eqs.(65) and (66)  $t^* = 2$  and  $\lambda = 1$ , we obtain

$$\alpha_2^*(1,0) = \alpha_1 \exp\left(2\frac{\alpha_2 - \alpha_1}{\alpha_1}\right) \tag{68}$$

and, respectively,

$$\alpha_2^*(1,1) = \frac{\alpha_1^2}{3\alpha_1 - 2\alpha_2}. (69)$$

As the final answer we set

$$\alpha_2^* = \frac{1}{2} \left[ \alpha_2^*(1,0) + \alpha_2(1,1) \right]. \tag{70}$$

Substituting into formulas (68) and (69) the numerical values from Eq.(57), we have

$$\alpha_2^*(1,0) = 1.1813, \qquad \alpha_2^*(1,1) = 1.1919.$$

Thence, Eq.(70) yields

$$\alpha_2^* = 1.1906.$$

This value of the critical screening parameter coincides with the result of numerical integration [11].

In conclusion, we have applied the self–similar renormalization theory [15-19] to calculating the energy and the critical screening parameter for the Schrödinger equation with the Yukawa potential. The calculated values are in good agreement with the results of numerical computation. This demonstrates that renormalization–group techniques can be successfully employed for solving the Schrödinger equation, not only with simple anharmonic potentials [28,29], but also for more realistic cases, such as the Yukawa potential that is often met in different physical problems. To get an accurate value of the critical screening parameter, we have used a method analogous to the algebraic self–similar renormalization [30]. We paid here attention mainly to the ground–state level, although the procedure we demonstrated is applicable to excited levels as well, but calculations become a little more cumbersome. We hope that the results we obtained are sufficient for showing the usefulness of renormalization–group techniques in quantum mechanics.

## Acknowledgement

We acknowledge financial support of the National Science and Technology Development Council of Brazil and from the University of Western Ontario, Canada..

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